High-Frequency Relic Gravitational Waves and Their Detection

L. P. Grishchuk

Cardiff University, United Kingdom

and Moscow State University, Russia

E-mail: grishchuk@astro.cf.ac.uk

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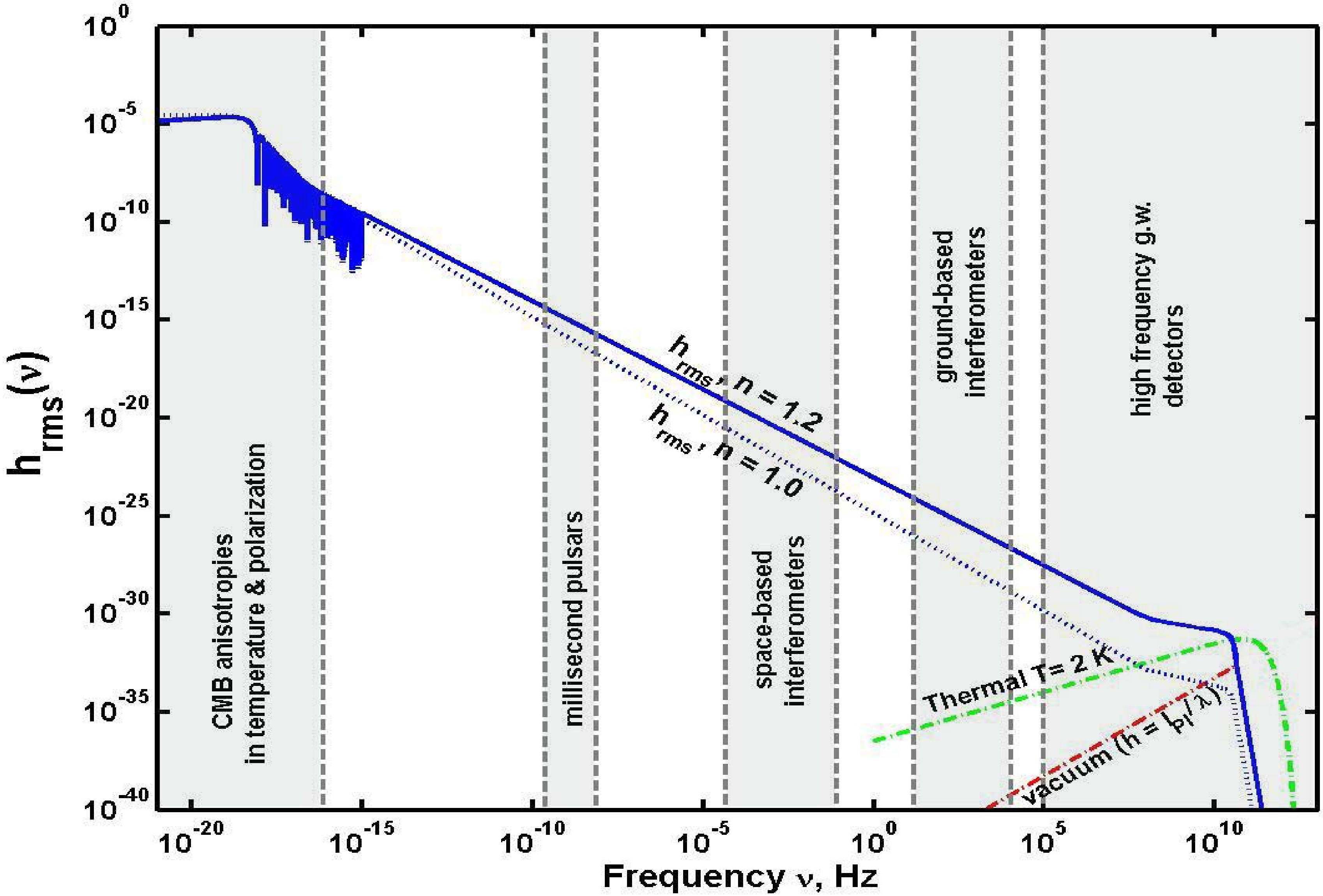
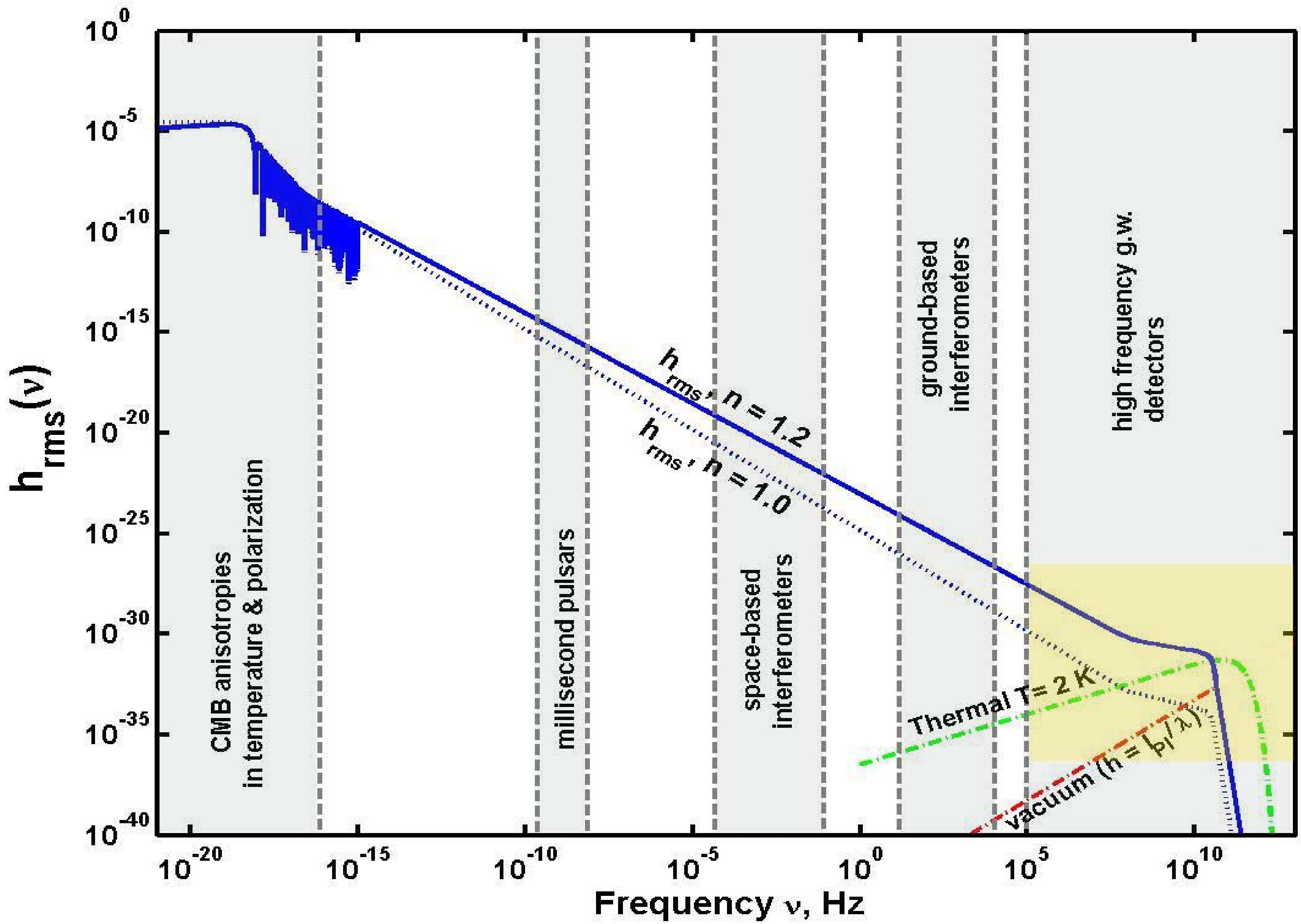
Abstract

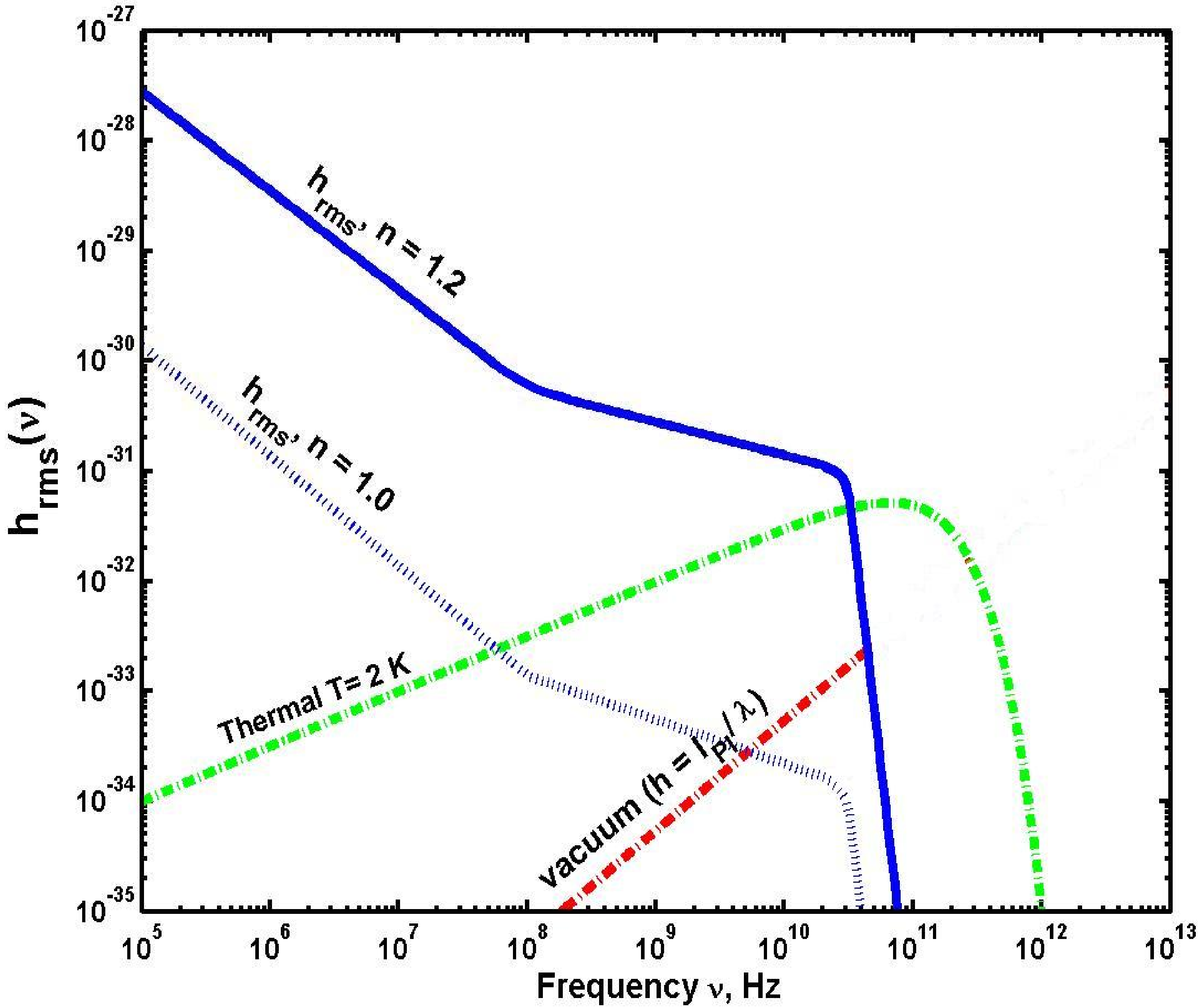
Most of this contribution is an updated summary of the theory on quantum-mechanical origin, physical properties and expected amplitudes of relic gravitational waves. In the second part of the presentation we will discuss the general principles of detection of high-frequency relic gravitational waves, and great difficulties on the way to this ambitious goal, as well as some new (old) experimental ideas.

Spectrum of relic gravitational waves



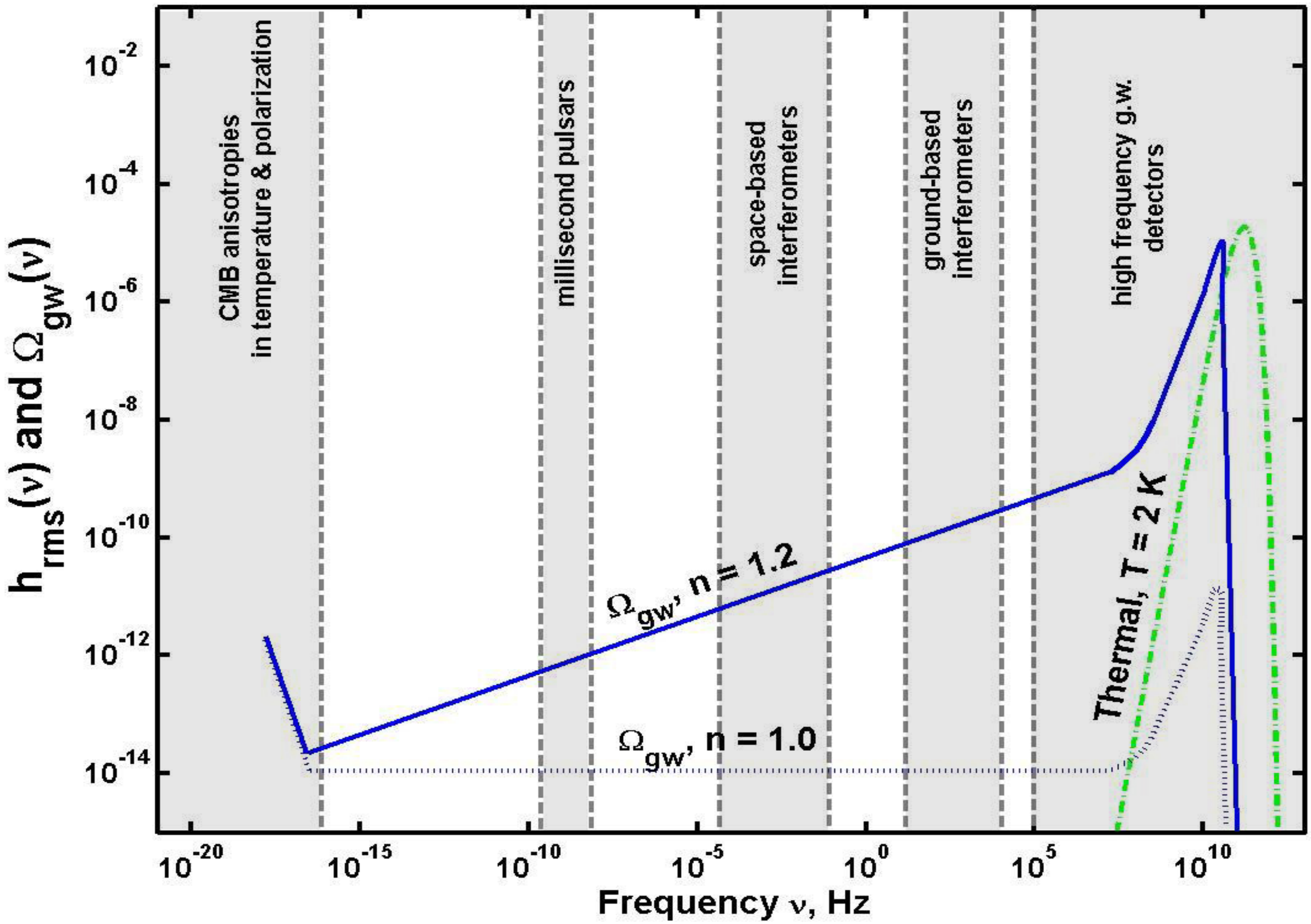


$h_{rms}(\nu)$
10^{0}
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-25}
10^{-30}
10^{-35}
10^{-40}
10^{-20}
10^{-15}
10^{-10}
10^{-5}
10^{0}
10^{5}
10^{10}
Frequency ν, Hz
CMB anisotropies
in temperature & polarization
millisecond pulsars
space-based
interferometers
ground-based
interferometers
high frequency g.w.
detectors
h_{rms}, $n = 1.2$
h_{rms}, $n = 1.0$
Thermal T= 2 K
vacuum ($h = l_{Pl}/\lambda$)



$h_{rms}(\nu)$
Frequency ν, Hz
h_{rms}, n = 1.2
h_{rms}, n = 1.0
Thermal T= 2 K
vacuum ($h = l_{Pl}/\lambda$)
10^{-27}
10^{-28}
10^{-29}
10^{-30}
10^{-31}
10^{-32}
10^{-33}
10^{-34}
10^{-35}
10^{5}
10^{6}
10^{7}
10^{8}
10^{9}
10^{10}
10^{11}
10^{12}
10^{13}

Energy density of relic gravitational waves $\Omega_{gw}(\nu) = \frac{\pi^2}{3} h^2(\nu) \left(\frac{\nu}{\nu_H}\right)^2$



Theory of cosmological (relic) gravitational waves

$$ds^2 = -c^2dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j = a^2(\eta)\left[-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right]$$

Spatial Fourier expansion of metric perturbations:

$$h_{ij}(\eta, \mathbf{x}) = \frac{\mathcal{C}}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} d^3\mathbf{n} \frac{1}{\sqrt{2n}} \sum_{s=1,2} \left[\overset{s}{p}_{ij}(\mathbf{n}) \overset{s}{h}_n(\eta) e^{i\mathbf{n}\cdot\mathbf{x}} \overset{s}{c}_{\mathbf{n}} + \overset{s}{p}_{ij}{}^*(\mathbf{n}) \overset{s}{h}_n{}^*(\eta) e^{-i\mathbf{n}\cdot\mathbf{x}} \overset{s}{c}{}^\dagger_{\mathbf{n}} \right],$$

Polarization tensors for gravitational waves (‘plus’ and ‘cross’ polarizations):

$$\overset{1}{p}_{ij}(\mathbf{n}) = l_i l_j - m_i m_j, \quad \overset{2}{p}_{ij}(\mathbf{n}) = l_i m_j + m_i l_j$$

Annihilation and creation operators satisfy the relationships:

$$[\mathbf{q}, \mathbf{p}] = i\hbar, \qquad \left[\mathbf{c}, \mathbf{c}^\dagger\right] = 1.$$

and act on the initial ground (vacuum) state of quantized gravitational waves

The Hamiltonian for each mode ‘n’ and each polarization state ‘s’

of gravitational waves

$$\mathbf{H}(\eta) = n\mathbf{c}^{\dagger}\mathbf{c} + \sigma\mathbf{c}^{\dagger 2} + \sigma^{*}\mathbf{c}^{2}$$

where the coupling function to the ‘pump’ field is $\sigma(\eta) = (i/2)(a'/a)$

Initial conditions: ground state (vacuum state) of the Hamiltonian:

$$\mathbf{c}|0\rangle = 0, \quad \langle 0|\mathbf{q}^{2}|0\rangle = \langle 0|\mathbf{p}^{2}|0\rangle = \frac{\hbar}{2}, \quad \Delta\mathbf{q}\Delta\mathbf{p} = \frac{\hbar}{2}.$$

Quantum-mechanical Schrodinger evolution transforms initial vacuum state into a strongly squeezed multiquantum vacuum state (stochastic collection of standing waves). This determines today’s amplitudes, power spectra, statistics, etc. of relic gravitational waves and primordial density perturbations; PRD ***42****, 3413 (1990)*

Rigorous definitions are based on quantum mechanics :

Mean-square amplitude of the field in the initial (Heisenberg) vacuum state:

$$\langle 0|\, h_{ij}(\eta,\mathbf{x}) h^{ij}(\eta,\mathbf{x})\, |0\rangle = \frac{\mathcal{C}^2}{2\pi^2}\int_0^\infty n^2 \sum_{s=1,2} |\overset{s}{h}_n(\eta)|^2 \frac{dn}{n}.$$

Power spectrum is a function of wavenumbers (and time):

$$h^2(n,\eta) = \frac{\mathcal{C}^2}{2\pi^2} n^2 \sum_{s=1,2} |\overset{s}{h}_n(\eta)|^2 = \frac{1}{2}\sum_{s=1,2} |\overset{s}{h}(n,\eta)|^2,$$

Statistical properties are determined by the statistics of squeezed vacuum states

Today's mean-square amplitude $\langle h^2\rangle = \int_0^\infty h^2(\nu)\frac{d\nu}{\nu}$

To simplify calculations, one can work with a `classical' version of the theory in which the field is characterized by classical random Fourier coefficients:

$$\langle \overset{s}{c}_{\mathbf{n}}\rangle = \langle \overset{s'*}{c}_{\mathbf{n}'}\rangle = 0, \quad \langle \overset{s}{c}_{\mathbf{n}} \overset{s'*}{c}_{\mathbf{n}'}\rangle = \langle \overset{s*}{c}_{\mathbf{n}} \overset{s'}{c}_{\mathbf{n}'}\rangle = \delta_{ss'}\delta^{(3)}(\mathbf{n}-\mathbf{n}'), \quad \langle \overset{s}{c}_{\mathbf{n}} \overset{s'}{c}_{\mathbf{n}'}\rangle = \langle \overset{s*}{c}_{\mathbf{n}} \overset{s'*}{c}_{\mathbf{n}'}\rangle = 0,$$

and no other statistical assumptions are being made.

Gravitational-wave equation, parametrically excited oscillator, physics of generation

$$\overset{s}{\mu}{}_n'' + \overset{s}{\mu}_n \left[n^2 - \frac{a''}{a} \right] = 0,$$

where

$$\overset{s}{\mu}_n\,(\eta) \equiv a(\eta)\,\overset{s}{h}_n\,(\eta) \quad \text{and} \quad {}' = d/d\eta = (a/c)d/dt$$

Conclusion: interaction with the 'pump' field and inevitable generation

of relic gravitational waves (Grishchuk, JETP,1974)

[Interesting comments: J. A. *Wheeler - "engine-driven cosmology", E. Schrodinger*

"alarming phenomenon" (he was right to worry about electromagnetic waves but

not about gravitational waves; for some details see PRD ***48*** *, 5581, 1993)*]

Relic gravitational waves allow us to make **direct inferences about the early universe Hubble parameter and scale factor** (`birth' of the Universe and its early dynamical evolution).

Superadiabatic (parametric) amplification: frequency of the oscillator $\omega^2 = g / l$ can be changed by the variation of length l of the pendulum (or strength g of the gravitational field) , $\ddot{x} + \omega^2(t)x = 0$

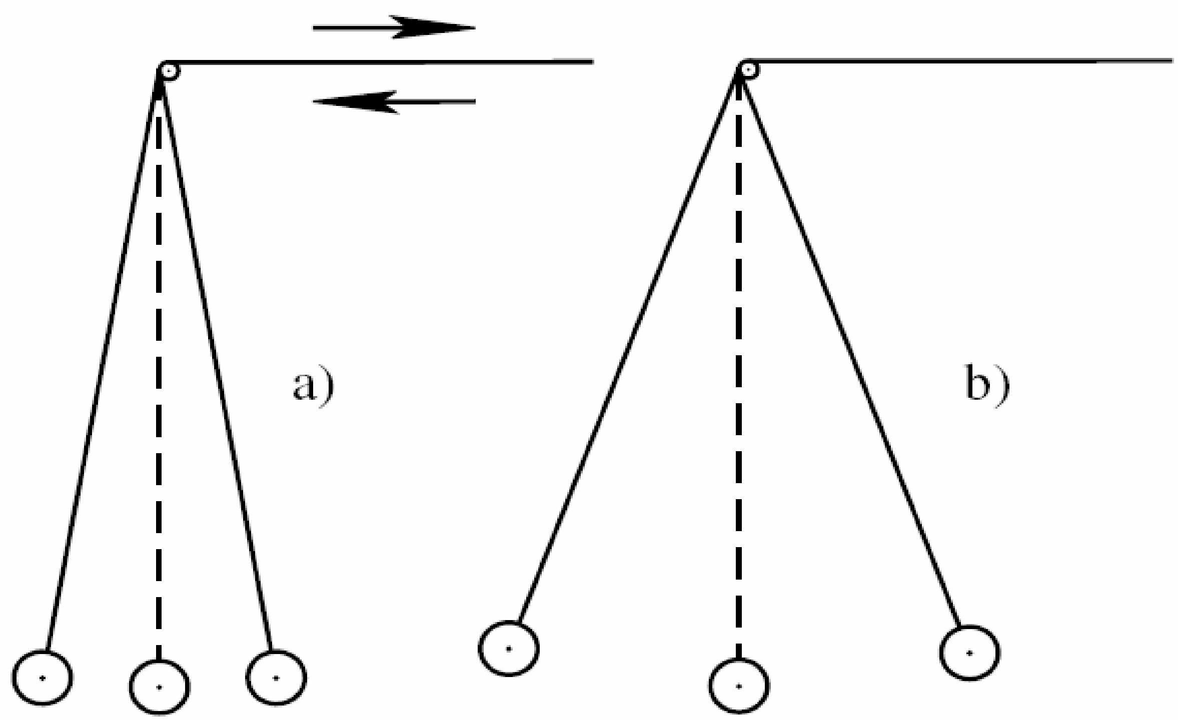


Fig. 1. Parametric amplification. a) variation of the length of the pendulum, b) increased amplitude of oscillations.

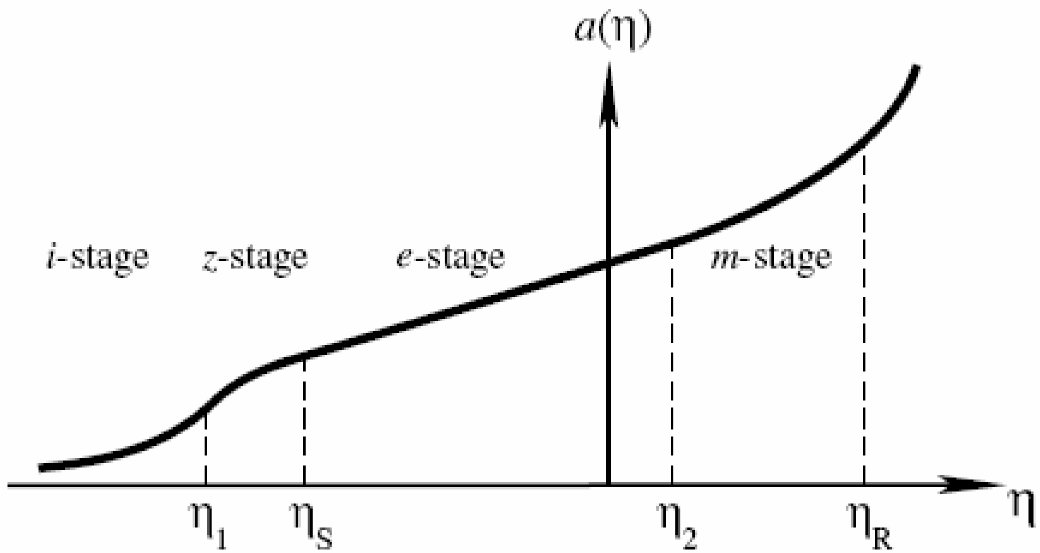


Fig. 4. Scale factor $a(\eta)$.

Engine-driven cosmology with some knowledge of some parts of the evolution

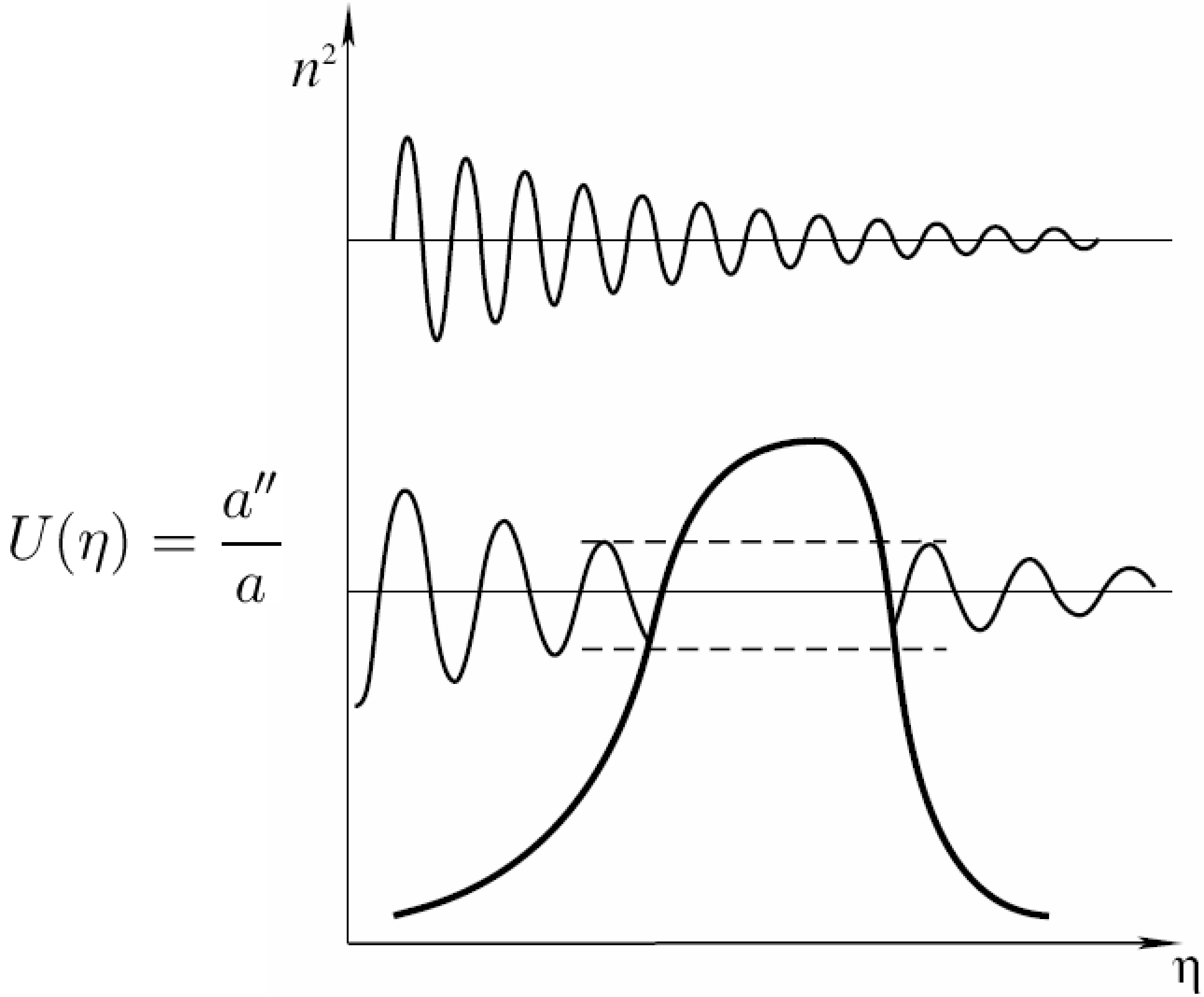


Fig. 3. Effective potential U(η).

Wavelength in comparison with the Hubble radius

(Gravitational wave equation: $$h'' + 2\frac{a'}{a}h' + n^2 h = 0$$)

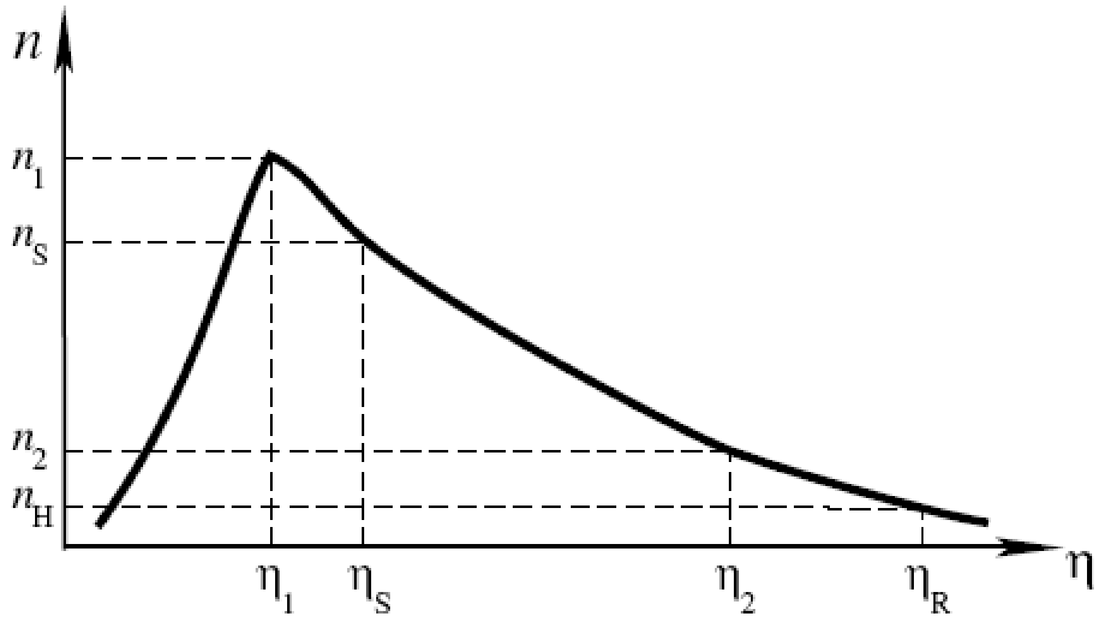
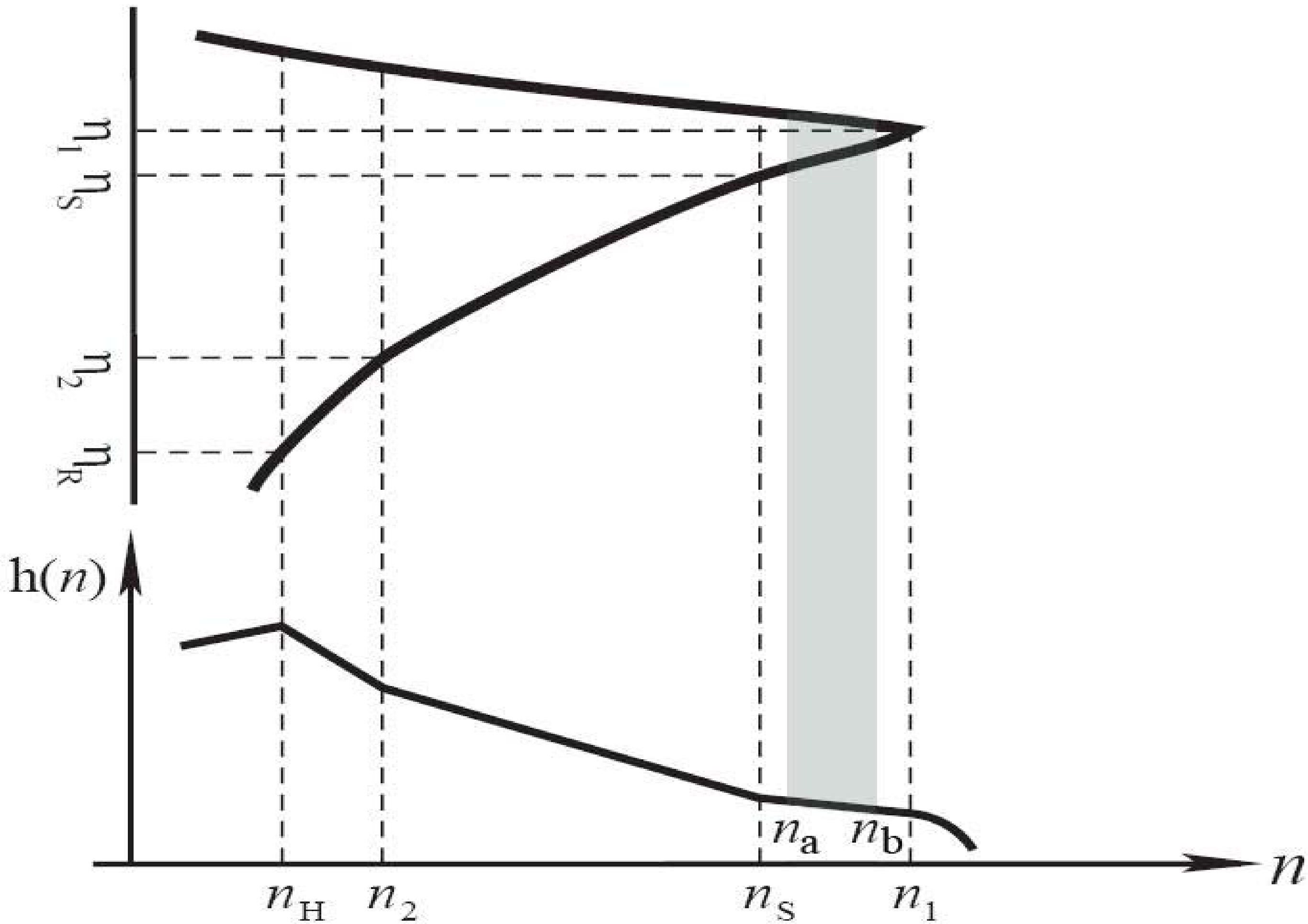
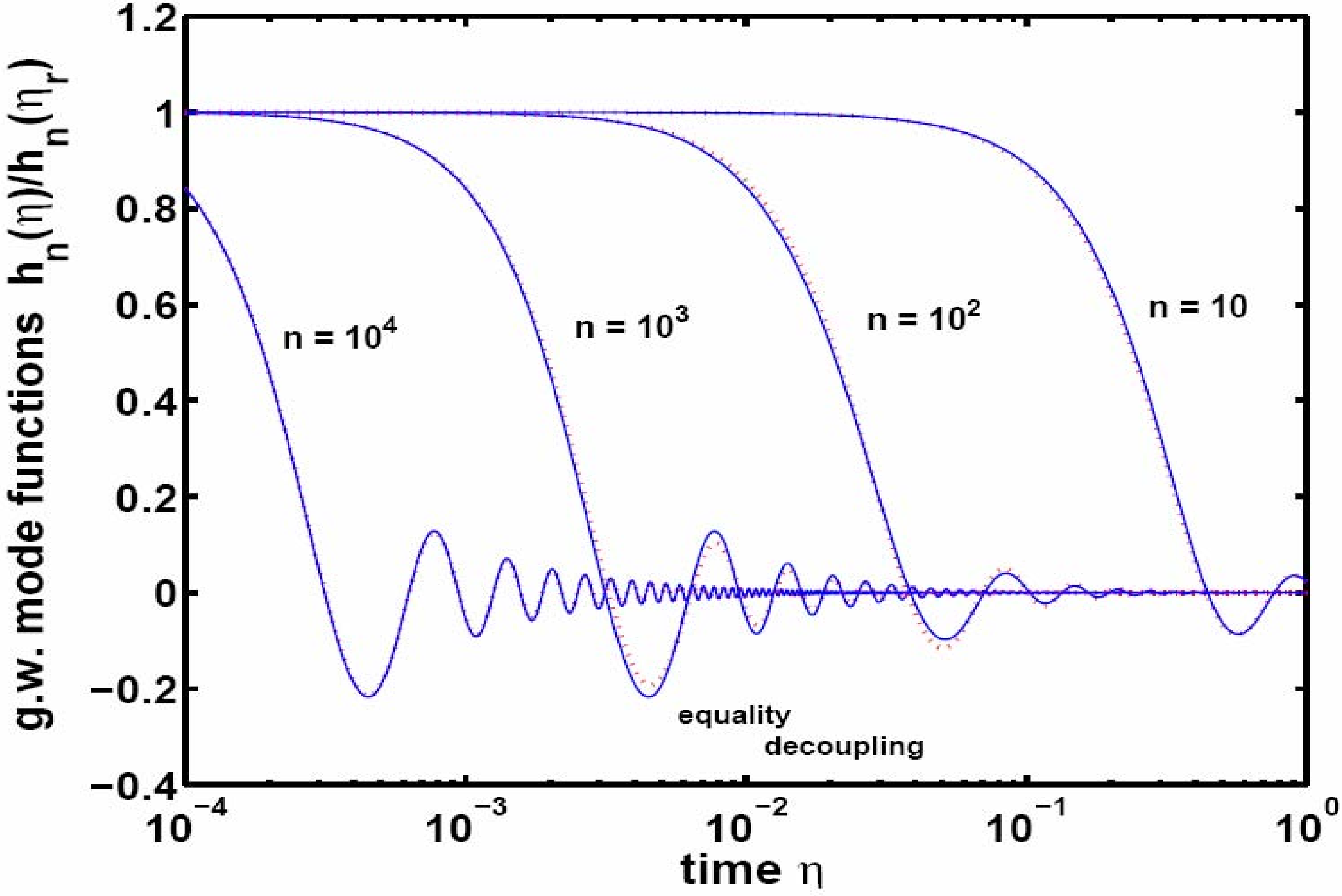


Fig. 5. Function $\frac{a'}{a}$ for the scale factor from Fig. 4.



The shape of the ‘barrier’ fully determines the shape of today’s spectrum



Evolution of mode functions (standing waves)

General principles of detection of a periodic gravitational wave signal

(Valid for electromagnetic and solid-state detectors,

see Sov. Phys. Uspekhi, vol. 20, pp. 319- 334, 1977)

(a) Conversion of a gravitational wave with frequency Ω on a static electromagnetic field $F_{\mu\nu(0)}$ with strength H.

(b) Interaction with oscillating electromagnetic mode of frequency $\dfrac{\Omega}{2}$ and characteristic strength E_0 .

(c) Interaction with the sum of a constant field H and a weaker oscillation E_0 on a natural frequency Ω

Signal in the detector

Case (a)

Change of field: $E_{(1)} \approx hQH$ change of energy: $\Delta\mathcal{E} \approx (hQ)^2\,\mathcal{E}$.

where $\mathcal{E} \propto H^2 V$

Case (b)

Change of field: $E_{(1)} \approx hQE_{(0)}$, change of energy: $\Delta\mathcal{E} \approx (hQ)\,\mathcal{E}$,

where $\mathcal{E} \propto E_{(0)}^2 V$

Case (c)

Change of field: $E_{(1)} \approx hQH$, change of energy $\Delta\mathcal{E} \approx (hQ)\,HE_{(0)}V$

'Quality factor' $Q \approx \Omega\tau^*$, volume $V \approx (c\pi/\Omega)^3$

Noise in the detector

Case (a)

Signal change of energy: $\Delta\mathcal{E} \approx (hQ)^2\,\mathcal{E}$.

Noise energy: $\mathcal{E}_{noise} \approx \hbar\Omega$

Case (b)

Signal change of energy: $\Delta\mathcal{E} \approx (hQ)\,\mathcal{E}$

Noise energy: $\mathcal{E}_{noise} \approx \sqrt{N}\hbar\Omega,\ N = \mathcal{E}/\hbar\Omega$

Case (c)

Signal change of energy $\Delta\mathcal{E} \approx (hQ)\,HE_{(0)}V$

Noise energy

$$\mathcal{E}_{noise} \approx \hbar\Omega\sqrt{E_{(0)}^2 V/\hbar\Omega}$$

Signal to noise ratio

In all three cases, assuming that the signal to noise ratio is 1, we arrive at approximately the same detectable amplitude:

$$h_{det} \approx \frac{1}{Q}\sqrt{\frac{\hbar\Omega}{\mathcal{E}}}$$

where $\mathcal{E}$ *is the appropriate (largest) energy involved in the detector*

The things are somewhat worse

in the case of a relic (stochastic) gravitational wave signal. The cross-correlation of outputs of two detectors allows the gain only in the proportion:

$$h_{det}^{relic} \approx \frac{1}{\sqrt{Q}}\sqrt{\frac{\hbar\Omega}{\mathcal{E}}}$$

Advantages and disadvantages of constant fields and oscillators

(for the same signal to noise ratio)

Constant fields in open space (Gertsenshtein effect):

Pro: Simplicity of manufacturing, strong fields achievable, low noise (1 quantum), broad-band response of the detector.

Contra: Unrealistically long interaction region required: $L = Q\lambda_{gw}$

Oscillators:

Pro: Long accumulation times, but compact systems; possibility (main advantage) to reduce noises below $\sqrt{N}$ *(for example, by squeezing); possibility to have broad-band detection in multi-mode systems (particularly, in a 'large crystal')*

Contra: Large natural noises, not less than $\sqrt{N}$ *If not specially reduced.*

(Copy of slide from HFGW1 with comments)

Detection of relic HFGW

$$h_{det} \approx \sqrt{\frac{\hbar\Omega}{\varepsilon}} \cdot \frac{1}{\sqrt{Q}}$$

If $\nu = 10^7$ Hz, $E \sim H \sim 3 \cdot 10^5$ gauss, $Q \approx 10^{13}$, $\tau^* = \dfrac{Q}{\omega}$,

then $h_{det} \approx 10^{-26}$ instead of $h_{signal} \approx 10^{-30}$ - a gap in 4 orders of magnitude.

Therefore, cross correlation of 2 or more detectors is of little help.
(useful, but not sufficient by itself)

Large composite antennae and/or
specially prepared quantum states are needed.

(seems to be a fundamental requirement not met until today)

New (old) idea: 'large crystal' (Grishchuk and Il'tchenko, 1984,

see also Sov. Phys. Uspekhi, **31**, 940, 1988)

Main result: in a multi-mode system with periodic boundary conditions (large crystal) the g.w. absorption cross section σ_n is independent of n , whereas in a system with free ends (like Weber's bar) σ_n decreases with n as $\sigma_n = \sigma_1 / n^2$ (see MTW, 1973, p.1035)

Therefore, a specially prepared 'large crystal' can be an effective absorber of high-frequency (relic) gravitational radiation , with frequencies $\nu_n = \frac{v_s}{l} n$

[The actual derivation assumed a 1-dimensional system (a long rod of length l) and a small size in comparison with relevant gravitational wavelengths, $l < \lambda_g$ The derivation should be generalized to a large 3-dimensional system.]

Equation for the oscillating rod under the action of a g.w. force:

$$-v_s^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{\tau}\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} = \frac{1}{2} h\omega^2 x \sin\omega t$$

Periodic boundary conditions: $u(-l/2, t) = u(l/2, t)$

In the regime of established oscillations:

$$u(x,t) = \sum_n B_n(t) \sin k_n x, \qquad n = 1, 2, 3, \ldots.$$

The energy of established oscillations is proportional to $\int F(x,t)u(x,t)dx$

Comparing with the g.w. energy flux, one finds the resonance integral:

$$\sigma_n \approx \frac{G}{c^3} M v_s^2, \quad \text{i.e.} \quad \sigma_n \approx \sigma_1$$

Total cross-section:

$\sigma_{tot} = n\sigma_n$ for 1-dimensional system

$\sigma_{tot} = n^3\sigma_n$ for 3-dimensional system

According to preliminary estimates, one can expect the production of a 1 new phonon in a cooled system of the total mass 10 kg. during the observation time of 1 day under the action of relic gravitational waves with the energy density of the order of the CMB energy density. This is as difficult to achieve as in the electromagnetic case, but with the advantage (disadvantage ?) of working with phonons rather than with the microwave photons.

Conclusions

There has been steady healthy progress since HFGW1 meeting, as demonstrated in some of the HFGW2 focus papers. The performed more detailed calculations and proposed new experimental schemes are useful and necessary. However, the fundamental obstacles, at the level of first principles, remain in force. We cannot hope to proceed without addressing them as the first and most urgent priority. Hopefully, this will be done in the coming years. All possibilities should be explored.